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# Resonant angular dependence in a weak magnetic field

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**Abstract.** Resonant tunnelling in two dimensions via a point impurity in a weak magnetic field is considered. We show that when the scattered electrons' energy is equal to the resonance one (which is the eigenenergy of the impurity), the shift of the tunnelling angle is half of the shift when the system is out of resonance. In particular, the incoming and outgoing angles depend on the impurity's location, and thus allow for its spectroscopy.

Experimental evidence of resonant tunnelling (RT) in optical systems has been known for hundreds of years (not by this name of course) [1]. That is, it was well known that the transmissivity of some medium could be greatly enhanced for a specific wavelength (or colour).

Soon after the birth of quantum mechanics, this concept of RT was adopted from the optical world to the quantum one. However, compared to optical systems, RT in the quantum systems is much more difficult to detect, because of the small wavelengths of the electrons. It was only in the last two decades, due to the improvements in nanostructure technology, that it became possible to fabricate a quantum system that can demonstrate an RT behaviour. Most of the experiments at the beginning were of one-dimensional (1D) nature [2]. Actually, in most of these experiments the two other dimensions were degenerated, e.g., two potential barriers separated by a potential well of layered semiconductors. RT in 1D was well investigated [3], and was found to be in good agreement with these experiments.

Besides these, there were other experiments (the first two articles of [4]) where their symmetry demands considerations of more than one dimension, e.g., where the potential well is not one dimensional but a potential defect in higher dimensions. There were some approximate models for RT through a point defect [5], which these experiments tried to confirm. The study of RT in higher dimensions goes on (the last two articles of [4]), but the angular dependence of the incident electrons in an RT process in 2D via an impurity received little attention.

The remarkable characteristics of the quantum Hall effect (QHE), that were demonstrated in 1980 [6], raise the interest in the combination of semiconductor heterostructures and a strong perpendicular magnetic field [7]. As a result, a few years later the influence of a strong magnetic field on a RT process was investigated [8]. Even though most of these experimental and theoretical works were of 1D nature, i.e., the RT through a 1D barrier was measured, the angular dependence of the RT electrons could not be ignored.

Marigliano Ramaglia *et al* showed [8] that when electrons are scattered from a potential barrier of 1D nature (without an impurity inside) in a strong perpendicular and localized magnetic field the tunnelling depends strongly on the angle of incidence of the incoming particles.

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In this paper we consider a model for RT in 2D via a point impurity in a weak magnetic field. It shows that the scattering amplitude, and thus the transmissivity of the barrier, exponentially increases when the electrons' energy equals the resonance energy of the impurity (otherwise, it is exponentially small). The amplitude also depends very strongly on the position of the impurity. It obtains its maximal value for a specific incoming and outgoing angles, and any deviation from these values will decrease it exponentially.

As a start, we choose a very simple model: a 2D electron gas scattered over an opaque stationary barrier (height $V$, width $2L$ in the $x$-direction and infinitely long in the $y$-direction) in a weak perpendicular magnetic field ($\boldsymbol{B} = -B\hat{\boldsymbol{z}}$).

Since we are interested in the scattering angle in a process of tunnelling through a magnetic field, we will confine the magnetic field to the area of the barrier. It should be emphasized here, that the existence of a confined magnetic field does not contradict any physical law. In fact, Maxwell equations allow for such a confinement so long as there is a current density at the boundaries (as in the case of a solenoid), which can be neglected in two dimensions. Many works (see the first two articles of [8]) and effects (see, for example, the Aharonov–Bohm effect [13]) used this fact, that a magnetic field can, in principle, be confined in space.

We can then write the stationary-state Schrödinger equation as

$$[(\hat{\boldsymbol{p}}_y + \boldsymbol{A}_y)^2 + \hat{\boldsymbol{p}}_x^2]\psi(\boldsymbol{r}) - (E - U(x))\psi(\boldsymbol{r}) = 0. \tag{1}$$

Hereinafter, we use the units $\hbar = 2m = -e = c = 1$ (Planck constant, the electron's mass and charge, and the velocity of light respectively). $\hat{\boldsymbol{p}}_{x,y}$ are the momentum operators, $E$ is the electrons' energy, $U$ is the potential:

$$U(x) \equiv \begin{cases} V & \text{for } -L < x < L \\ 0 & \text{otherwise.} \end{cases}$$

In the following, the Landau gauge is chosen:

$$A_y \equiv \begin{cases} -BL & x < -L \\ Bx & -L < x < L \\ BL & x < L. \end{cases}$$

The eigenfunction of (1) has the form

$$\psi(\boldsymbol{r}) = \eta(x)\exp(\mathrm{i}\kappa y).$$

Outside the barrier, the solution is trivial: if the incoming plane wave has a momentum $\boldsymbol{k}(\boldsymbol{k} \equiv \sqrt{E}[\cos\chi\,\hat{\boldsymbol{x}} + \sin\chi\,\hat{y}])$, i.e., with an incoming angle $\chi$, in our gauge the wave function is

$$\psi_{inc}(\boldsymbol{r}) = \exp(\mathrm{i}[\boldsymbol{k}\cdot\boldsymbol{r} - BLy]).$$

We are interested in small angles, thus,

$$\psi_{inc}(\boldsymbol{r}) = \exp(\mathrm{i}[kx + k\chi y - BLy]) \tag{2}$$

where $k \equiv \sqrt{E}$. Similarly, the scattered function looks like

$$\psi_{sc}(\boldsymbol{r}) = T\exp(\mathrm{i}[kx + k\theta y + BLy])$$

where $T$ is the transmission coefficient, and $\theta$ is the outgoing angle.

However, inside the barrier, we can rewrite equation (1):

$$\frac{\partial^2\eta}{\partial\xi^2} + \left(\varepsilon' - \frac{\xi^2}{4}\right)\eta = 0$$

where $\varepsilon' \equiv (E-U)/2B$, $\xi \equiv \sqrt{2B}(x - x_0)$ and $x_0 \equiv \kappa/B$.

The solutions of this equation are, of course, the parabolic cylinder functions $D_v(\pm\xi)$ (here we have exploited the fact that the magnetic field is very weak, i.e., $BL^2 \ll 1$. This allows us, along with the extreme opaqueness of the barrier ($UL^2 \gg 1$) to use the approximation $D_v(\pm\xi) \approx \exp(\mp\beta\xi \mp \xi^3/24\beta)$ (where $\beta \equiv \sqrt{|\varepsilon'|} = \sqrt{(U-E)/2B}$).

By matching the solutions and their derivatives at the boundary of the barrier, one finds that the scattered function $\psi_{sc}(\boldsymbol{r})$ (for $x \to \infty$) has the following form

$$\psi_{sc}(\boldsymbol{r}) = T\exp\{\mathrm{i}[kx + k(\chi - 2BL/k)y + BLy]\} \tag{2a}$$

where the transmission coefficient is

$$T \approx \exp\left\{-2Lp\left[1 + \left[\frac{k\chi - BL}{p}\right]^2\right]\right\} \tag{2b}$$

where $p \equiv \sqrt{U-E}$ (notice the difference between $p$ and $k \equiv \sqrt{E}$).

Expression (2*a*) suggests that the scattered angle is shifted by

$$\chi - \theta = 2BL/k$$

and expression (2*b*) suggests that maximal transmissivity through the barrier is achieved for

$$\chi = BL/k. \tag{2c}$$

But (2*b*) also implies that the transmissivity is always (for all incoming angles) exponentially small.

In the next step we add an impurity (or a defect) to the barrier. The stationary-state Schrödinger equation can then be written as

$$[(\hat{\boldsymbol{p}}_y + \boldsymbol{A}_y)^2 + \hat{\boldsymbol{p}}_x^2]\psi(\boldsymbol{r}) - (E - U(x))\psi(\boldsymbol{r}) = D(\boldsymbol{r} - a\hat{\boldsymbol{x}})\psi(\boldsymbol{r}) \tag{3}$$

where $D(\boldsymbol{r})$ is the impurity's potential and $a$ is the distance from the impurity to the centre of the barrier (see figure 1(b)).

Equations like equation (3) are complicated, and there is no general method to solve them. One cannot use perturbation theory, because the problem includes tunnelling. The WKB approximation is also inapplicable since it can be used only when the electron's wavelength is larger then the problem's length parameters. The last requirement certainly does not hold because of the pointlike nature of the defect.

Within the following lines, we show that by using an impurity D function (IDF, see [9]) as the defect potential, the solution to equation (3) will merely be the Green function of the same equation but without the defect.

Even though the IDF represents a point impurity, the implications of the solution are more general, since it can be generalized to a short-range scatterer [12].

Since the symmetry of the barrier is Cartesian, it is convenient to use the extreme anisotropy case of a point potential $D(\boldsymbol{r}) = W\delta(x)v(y/\rho)$ (where $\rho$ is the impurity size in the $y$-direction, $\delta$ is the Dirac delta function, $W$ is a function of $\rho$ only and $v(z)$ is a function that decays at $z \sim 1$). By taking the limit $\rho \to 0$, one can choose the following IDF as the potential defect (see also [9]):

$$D(\boldsymbol{r}) \equiv \lim_{\rho\to 0} \frac{2\pi^{1/2}}{\rho\ln(\rho_0/\rho)}\delta(x)\exp[-(y/\rho)^2].$$

The delta function can, of course, be written as

$$\delta(x) = \lim_{\rho\to 0} \exp[-(x/\rho)^2]/\rho\pi^{1/2}.$$

Thus, the size of the defect is $\rho$ in both dimensions, which means that its zero point energy is $\approx 1/\rho^2$. Since $|D| \ll 1/\rho^2$, when $\rho \to 0$, the potential defect is like an infinitely shallow

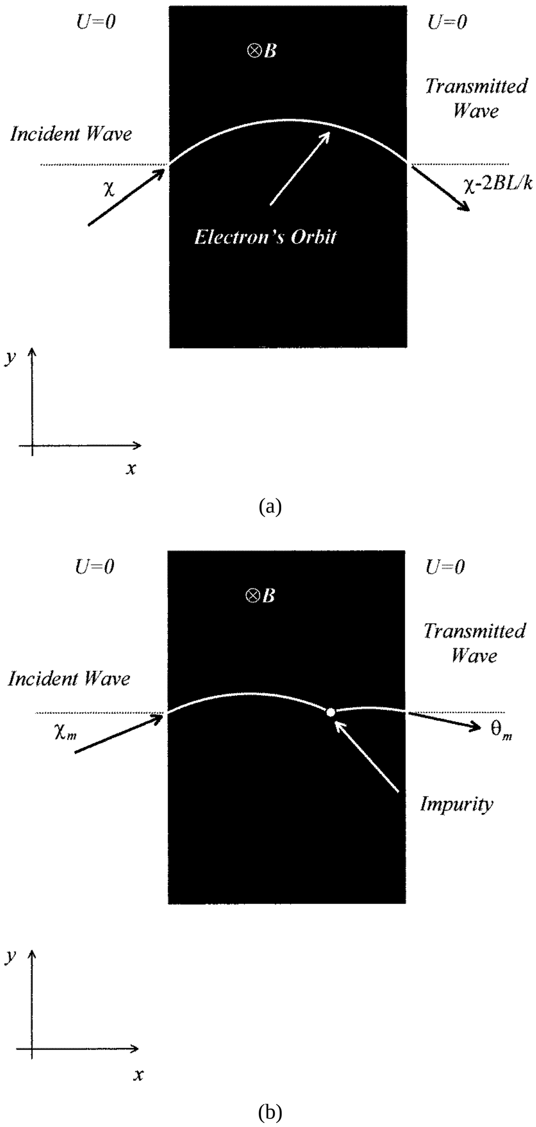


**Figure 1.** (a) The direction of the incoming electrons is shifted when scattered over the barrier in the presence of a weak magnetic field. But the transmissivity is exponentially small. (b) When the energy of the incoming electrons is equal to the resonance one, the transmissivity exponentially increases, and attains its maximal value for $\chi = \chi_m \equiv (L+a)B/2k$ and $\theta = \theta_m \equiv -(L-a)B/2k$.

well. For such a well, there is a bound eigenstate with the finite Bohr radius $\rho_B \approx 0.75\rho_0$, and an eigenenergy $E_0 \sim -1/\rho_B^2$ [9, 10]. Since its eigenfunction exponentially decays at $\rho_B$, while $D$ decays at $\rho$ (i.e., the potential decays infinitely faster than its eigenfunction), in the region of the defect the wave function can be considered as a constant, as if it were a delta function [10] (this is also valid for extended states).

Then, equation (3) can be rewritten

$$[(\hat{\boldsymbol{p}}_y + \boldsymbol{A}_y)^2 + \hat{\boldsymbol{p}}_x^2]\psi(\boldsymbol{r}) - (E - U(x))\psi(\boldsymbol{r}) = D(\boldsymbol{r} - a\hat{\boldsymbol{x}})\psi(a\hat{\boldsymbol{x}}). \qquad (4)$$

The solution for such an equation comes directly from its Green function. Then, the scattered eigenfunction $\psi^+_{sc}(\boldsymbol{r})$ is [11]

$$\psi^+_{sc}(\boldsymbol{r}) = \psi_{sc}(\boldsymbol{r}) - \frac{G^+(\boldsymbol{r}, \boldsymbol{r}_0)\psi_{sc}(\boldsymbol{r}_0)}{1 + \int \mathrm{d}\boldsymbol{r}'\, G^+(\boldsymbol{r}_0, \boldsymbol{r}')D(\boldsymbol{r}' - \boldsymbol{r}_0)} \int \mathrm{d}\boldsymbol{r}''\, D(\boldsymbol{r}'' - \boldsymbol{r}_0) \qquad (5)$$

where $\boldsymbol{r}_0 \equiv a\hat{\boldsymbol{x}}$ is the impurity's position, $\psi_{sc}$ is the incoming plane wave (or any other solution of the homogenous equation) and $G^+(\boldsymbol{r}, \boldsymbol{r}')$ is the 'outgoing Green function' of equation (4). Although (5) is the exact solution of the problem, it is rather complicated, because of the problematic symmetry of the problem. However, in the case of a very opaque barrier (high and wide) and a weak magnetic field, for which

$$B^{-1/2} \gg L \gg E^{-1/2} > V^{-1/2}$$

the leading approximation yields an explicit analytical expression.

The calculations of the Green function are very tedious; thus we will not present them here except for the main points.

It is inconvenient to calculate the Green function from the eigenfunctions. Instead, we prefer to use the helpful relation

$$G^+(\boldsymbol{r}, \boldsymbol{r}_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k\, G^+_1(x, x_0; E - k^2)\, \mathrm{e}^{\mathrm{i}ky}. \qquad (5a)$$

In this equation $G^+_1(x, x_0; E)$ is the 'outgoing Green function' of the one-dimensional equation:

$$[\hat{p}^2_x + (k + A_y)^2]\psi(x) - (E - U)\psi(x) = 0. \qquad (5b)$$

$G^+_1$ can easily be calculated, since the solutions of equation (5*b*) are known exactly (with the $D_\upsilon$ as in equation (1)). Thus, by using the weak field approximation, which was used in (2*b*), $G^+$ can be calculated from (5*a*).

If we rewrite equation (5) as

$$\psi^+_{sc}(\boldsymbol{r}) = \psi_{sc}(\boldsymbol{r}) + \frac{f}{r^{1/2}} \exp(\mathrm{i}kr + \mathrm{i}BLy)$$

for $r \equiv |\boldsymbol{r}| \to \infty$, then the scattering amplitude $(f)$ has the form:

$$f(\chi, \theta, E) = C \exp\left[ -2pL - \frac{L+a}{2p}\left(k\chi - \frac{L+a}{2}B\right)^2 - \frac{L-a}{2p}\left(k\theta + \frac{L-a}{2}B\right)^2 \right] \times [\ln(p^2/E_0) + 2\mathrm{i}\Delta]^{-1} \qquad (6)$$

where $\chi$ is the incident angle, i.e., $\boldsymbol{k} \equiv (\hat{\boldsymbol{x}} \cos\chi + \hat{\boldsymbol{y}} \sin\chi)$, $\theta$ is the scattered angle, i.e., $\boldsymbol{r} \equiv r(\hat{\boldsymbol{x}} \cos\theta + \hat{\boldsymbol{y}} \sin\theta)$, $C \approx 1$ is a complex function that depends slowly on the angles $\theta$, $\chi$ and on the energy $E$. The resonance width is exponentially small $\Delta \approx L^{-1/2}U^{-1/4} \exp[-2p(L-|a|)]$, and as was noted $p^2 \equiv V - E$.

From equation (6) one can learn the following:

(A) A resonance occurs for $E \approx E_r \equiv V - E_0$. In that case, the real part of the denominator vanishes, and the amplitude exponentially increases by the factor $\Delta^{-1}$.

(B) The width of the resonance is exponentially small, i.e., $2E_0\Delta$. The dependence of the resonance energy and its width on the magnetic field is negligible.

(C) The incoming and outgoing angles, for which the amplitude is maximal, depend on the magnetic field and on the impurity's position in the barrier: $f$ attains its maximal value for an incoming angle $\chi_m \equiv (L + a)B/2k$, and for an outgoing one $\theta_m \equiv -(L - a)B/2k$.

In the following we present a classical interpretation of this effect.

The favourite orbit (the one with the maximal amplitude) is constructed from two arcs: The first arc begins at the left side of the barrier and ends at the impurity's location. Its centre of orbit is at $-(L-a)/2$. The second arc begins at the impurity's location and ends at the right side of the barrier, with a centre of orbit at $(L+a)/2$ (see figures 1(b)).

The difference between these angles, i.e.,

$$\chi_m - \theta_m = LB/k$$

is half of the difference between the incoming and outgoing angles when the impurity is missing (equations (2*c*), see figure 1(a)). This may sounds strange, because this difference does not depend on the impurity's location. It is valid even when the impurity is very close to the edges of the barrier, while in that case one might expect to regain the previous result (where the impurity is absent).

This effect is a quantum mechanical one (since it is a resonant tunnelling effect); however, since the magnetic field is very weak, the electron's orbit may have a quasi-classical interpretation:

Inside the barrier, the particle 'prefers' to minimize its path (with the constraint of moving on a circle with a radius $k/B$). Thus, when the impurity is absent, the minimized path is a symmetrical arc (see figure 1(a)), and then the incoming and outgoing angles are $BL/k$ and $-BL/k$ respectively. When the impurity is present, and the system is at resonance, the particles must pass through the impurity. The rest of the path is again two symmetrical arcs (see figure 1(b)), and thus the incoming and outgoing angles are $(L+a)B/2k$ and $-(L-a)B/2k$ respectively.

Let us evaluate these angles. Our derivation is correct for $\sqrt{U-E_r} \gg L^{-2} \gg B$. Then, in an ordinary experiment the values $L \approx 0.1$ $\mu$m and $B \approx 10^{13}$ m$^{-2}$ ($B \approx 10^{-3}$ T) will lead

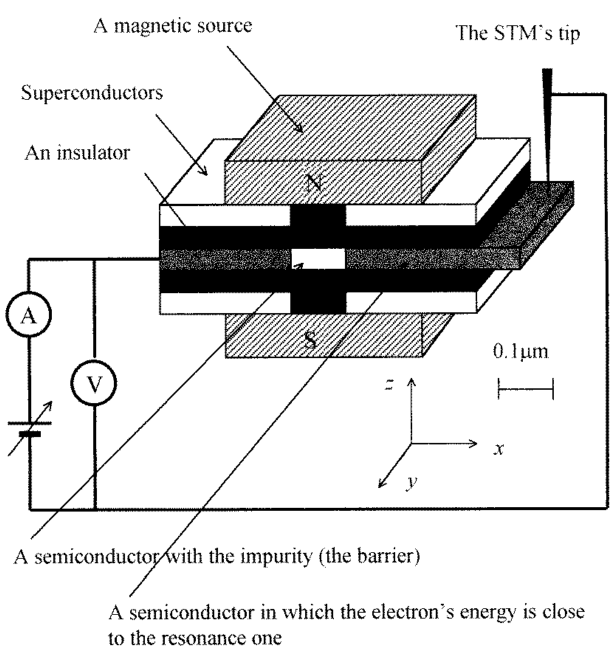


**Figure 2.** A qualitative experiment with an STM to measure the scattered angle of the electrons, which tunnel through a 2D point impurity in a weak magnetic field.

to $\chi$, $\theta \approx 10^{-4} E_{eV}^{-1/2}$ ($E_{eV}$ is the electrons' energy in electron Volts). Thus, if the current is measured 1 $\mu$m from the impurity (in figure 2 it is $\approx$0.5 $\mu$m), the orbit deviation in the $y$-direction is $\approx$1 Å $E_{eV}^{-1/2}$. One can place the detector in a potential energy $U = V_2$, for which $E_{eV} = E_r - V_2 \approx 0.01$ eV, and then the orbit's deviation will be $\approx$10 Å. This value can be measured with a scanning tunnelling microscope (STM). A qualitative experiment with an STM is shown in figure 2. The barrier is represented as a doped semiconductor (say AlGaAs or $SiO_2$), which allows at the Fermi level for evanescent modes. The undoped semiconductors (say GaAs or Si) on its two sides allow for electron propagation. The matching between the electron's Fermi energy and the impurity's resonance energy can be obtained in numerous ways, one of which is that the imperfection (the impurity) dimensions can be engineered to meet the electron's Fermi wavelength, while small energy modifications can be adjusted by the voltage source (of course, that will distort the barrier but will have a negligible effect on the electron's orbit). Another option is to create a multiple-imperfection barrier, in which case only the resonant impurities (the ones where their resonance energy corresponds with the Fermi one) will contribute to the effect.

The orbit's deviation can, of course, be used to detect the location of the impurity: When the current through the STM's tip is maximal, while it is moving only in the $y$-direction, and the magnetic field is absent, the $y$-coordinate of the impurity is determined. Then the magnetic field is turned on. The new position of the tip, for which the current is maximal, determines the shift of the scattered angle.

To summarize, we consider a model for RT in 2D via a point impurity in a weak magnetic field. This model shows that when the electrons' energy is equal to the resonance one, the transmissivity of the barrier exponentially increases. The transmissivity depends exponentially on both the incoming and outgoing angles, and attains its maximal values for specific ones. Since these angles depend on the impurity's location, this effect can be used to map impurities in a potential barrier. It also emphasizes that the impurity pins the electron and deviates its orbit. A qualitative experiment is also suggested.

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